On quasi-free dynamics on the resolvent algebra

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May 16 2014

Abstract

The resolvent algebra is a new C^*-algebra of the canonical commutation relations given by Buchholz-Grundling. We study analytic properties of quasi-free dynamics on the resolvent algebra. Subsequently we consider a supersymmetric quasi-free dynamics on the graded C^*-algebra made of a Clifford (fermion) algebra and a resolvent (boson) algebra. We establish an infinitesimal supersymmetry formula upon the GNS Hilbert space for any regular state satisfying some mild requirement which is standard in quantum field theory.

Contents

1 Introduction

The resolvent algebra is a C^*-algebra generated by resolvents of canonical operators for a boson quantum field. Its structure and properties have been clarified in [1, 2].

In this note we study dynamics on the resolvent algebra. We show analytic properties of general quasi-free dynamics on the resolvent algebra in terms of the Gel'fand-Naimark-Segal (GNS) representation for regular states. Although these results can be expected from the corresponding quasi-free dynamics on the Weyl algebra, see [2, Corollary.4.4],

we have to treat carefully resolvent operators in connection with boson and fermion field operators. Furthermore we provide new insights on supersymmetric dynamics.

We consider a simplified free supersymmetric model given in [1]. It is formulated on the fermion-boson system which is given by the C^* tensor-product of a Clifford algebra and a resolvent algebra. We establish an infinitesimal supersymmetry formula upon the GNS Hilbert space for any regular state of the fermion-boson C^*-system satisfying some natural assumption. This assumption is expected to hold for general states in quantum field theory, such as vacuum states and excited states.

We shall compare our new formula with the corresponding one shown in [1]. Our supersymmetry formula is based on the strong topology of the GNS Hilbert space and it is valid for any regular state satisfying the mentioned assumption. On the other hand, the similar supersymmetry formula stated in [1] relies on the weak topology (i.e. expectation values of a state), and it makes sense only for supersymmetric states. Precisely, in [1, Theorem.5.8] the weak supersymmetry formula was given for an unbounded supersymmetric functional.

There are several essential improvements produced by our new supersymmetry formula. It has been shown in [3] that our strengthened supersymmetry formula yields several consequences. Among others, the supercharge operator upon the GNS Hilbert space for any regular supersymmetric state is self-adjoint. This statement is proved under a very general setting and applies to the present quasi-free model.

2 Resolvent algebra

We recall the resolvent algebra and its basic properties, see [1, 2] for the detail.

2.1 Definition

Let (X,σ) denote a symplectic space. Let $\mathcal{R}_0$ denote the universal unital $*$-algebra generated by $\big\{R(\lambda,\, f) \mid f\in X,\ \lambda\in\mathbb{R}\setminus\{0\}\big\}$ satisfying the following relations [1, Definition 3.2] [2, Definition 3.1]: For $f,g\in X$ and $\lambda,\mu\in\mathbb{R}\setminus\{0\}$

$$\begin{aligned}
R(\lambda,\, 0) &= -\frac{i}{\lambda}1,\\
R(\lambda,\, f)^* &= R(-\lambda,\, f),\\
R(\lambda,\, f) &= \frac{1}{\lambda}R\big(1,\, \tfrac{f}{\lambda}\big),\\
R(\lambda,\, f)-R(\mu,\, f) &= i(\mu-\lambda)R(\lambda,\, f)R(\mu,\, f),\\
[R(\lambda,\, f),\ R(\mu,\, g)] &= i\sigma(f,\, g)R(\lambda,\, f)R(\mu,\, g)^2R(\lambda,\, f),\\
R(\lambda,\, f)R(\mu,\, g) &= R(\lambda+\mu,\, f+g)\big\{R(\lambda,\, f)+R(\mu,\, g)+i\sigma(f,g)R(\lambda,\, f)^2R(\mu,\, g)\big\},
\end{aligned}\tag{2.1}$$

where $\lambda+\mu\neq 0$ is required in the last equation. These abstractly characterize the set of resolvents associated with a neutral boson field over (X,σ). Assume that the symplectic space (X,σ) is non-degenerate.

By endowing a C^*-seminorm with $\mathcal{R}_0$ we make a unital C^*-algebra denoted as $\mathcal{R}\big(X,\,\sigma\big)$. This C^*-algebra will be called *the resolvent algebra* over $(X,\,\sigma)$. Note that there are some natural C^*-seminorms on $\mathcal{R}_0$, see [2, Remark 3.5]. Our argument below does not depend on the choice of such C^*-seminorms. By [1, Theorem 3.4(ii)] the norm of resolvents is

determined by

$$\|R(\lambda,\ f)\| = \frac{1}{\lambda} \quad \text{for all } f \in X \ \text{ and } \lambda \in \mathbb{R}\setminus\{0\}. \tag{2.2}$$

Take the following norm-dense the dense $*$-subalgebra of $\mathcal{R}\big(X,\ \sigma\big)$ which is algebraically isomorphic to $\mathcal{R}_0$:

$$\mathcal{R}_0\big(X,\ \sigma\big) := *\text{-alg}\big\{R(\lambda,\ f) \mid f \in X,\ \lambda \in \mathbb{R}\setminus\{0\}\big\} \subset \mathcal{R}\big(X,\ \sigma\big). \tag{2.3}$$

2.2 Regular representations and boson field operators

Let π be a $*$-representation of the resolvent algebra $\mathcal{R}\big(X,\ \sigma\big)$ upon a Hilbert space $\mathscr{H}_\pi$. If

$$\operatorname{Ker} \pi\big(R(1,\ f)\big) = 0 \text{ for every } f \in X, \tag{2.4}$$

then $(\pi,\ \mathscr{H}_\pi)$ is called *regular*. Let ω be a state of $\mathcal{R}\big(X,\ \sigma\big)$. Take its GNS representation denoted as $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$. If the representation $(\pi_\omega,\ \mathscr{H}_\omega)$ of $\mathcal{R}\big(X,\ \sigma\big)$ is regular, then the state ω is called *regular*. The above definitions are due to [1, Definition 3.5] [2, Definition 4.3].

For any regular representation $(\pi,\ \mathscr{H}_\pi)$ of $\mathcal{R}\big(X,\ \sigma\big)$ we can define boson field operators by

$$j_\pi(f) := iI - \pi\big(R(1,\ f)\big)^{-1} \ \text{ on } \operatorname{Dom}(j_\pi(f)) = \operatorname{Ran}\left(\pi(R(1,\ f))\right) \subset \mathscr{H}_\pi \ \text{ for all } f \in X, \tag{2.5}$$

where I is an identity operator on $\mathscr{H}_\pi$. From (2.5) (2.1) for each $\lambda \in \mathbb{R}\setminus\{0\}$

$$j_\pi(f) = i\lambda I - \pi\big(R(\lambda,\ f)\big)^{-1} \ \text{ on } \operatorname{Dom}(j_\pi(f)) = \operatorname{Ran}\left(\pi(R(1,\ f))\right) \ \text{ for all } f \in X. \tag{2.6}$$

Conversely,

$$\pi\big(R(\lambda,\ f)\big) = \big(i\lambda I - j_\pi(f)\big)^{-1} \text{ on } \mathscr{H}_\pi. \tag{2.7}$$

We collect some basic properties of the field operators induced by a regular representation $\big(\pi,\ \mathscr{H}_\pi\big)$ without proof. In [1, Theorem 3.6(i)] [2, Theorem.4.2(i)] it has been shown that for every $f \in X$

$$j_\pi(f) \ \text{ is a self-adjoint operator on } \mathscr{H}_\pi, \tag{2.8}$$

and that

$$\pi\left(R(\mu,\ g)\right) \operatorname{Dom}\big(j_\pi(f)\big) \subset \operatorname{Dom}\big(j_\pi(f)\big) \ \text{ for all } g \in X \text{ and } \mu \in \mathbb{R}\setminus\{0\}. \tag{2.9}$$

From (2.9) it follows that any operator in $\pi\left(\mathcal{R}_0\big(X,\ \sigma\big)\right)$ preserves the domain of $j_\pi(f)$:

$$\pi\left(\mathcal{R}_0\big(X,\ \sigma\big)\right) \operatorname{Dom}\big(j_\pi(f)\big) \subset \operatorname{Dom}\big(j_\pi(f)\big). \tag{2.10}$$

It has been shown in [1, Theorem 3.6(vii)] [2, Theorem. 4.2(vii)] that for every $f, g \in X$ and $\lambda \in \mathbb{R}\setminus\{0\}$,

$$\left[j_\pi(f),\ \pi\left(R(\lambda,\ g)\right)\right] = i\sigma(f, g)\pi\left(R(\lambda,\ g)^2\right) \ \text{ on } \ \operatorname{Dom}\big(j_\pi(f)\big). \tag{2.11}$$

The following asymptotic formula is given in [1, Theorem 3.6(ii)] [2, Theorem 4.2(ii)]:

$$i\lambda\pi\big(R(\lambda,\ f)\big) \longrightarrow I \ \text{ as } \ \lambda \to \infty \ \text{ in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi) \ \text{ for every } f \in X. \tag{2.12}$$

By the third equation of (2.1), this is equivalent to

$$i\pi\big(R\big(1,\ \tfrac{f}{\lambda}\big)\big) \longrightarrow I \ \text{ as } \ \lambda \to \infty \ \text{ in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi) \ \text{ for every } f \in X. \tag{2.13}$$

Remark 2.1. There is a bijection between regular representations of $\mathcal{R}\big(X,\,\sigma\big)$ and those of the Weyl algebra over the same $(X,\,\sigma)$. This is given as [2, Corollary.4.4].

Remark 2.2. The notion of regular representations given above can be generalized to a larger system that contains $\mathcal{R}\big(X,\,\sigma\big)$ as its subalgebra, for example, the fermion-boson system which will be discussed in Sect.5.

The following fact will be useful for later discussion. (The point is that the domain of this equality is not merely the dense subspace $\mathrm{Dom}\big(j_\pi(f)\big)$ but the total Hilbert space $\mathscr{H}_\pi$.)

Lemma 2.3. *Let* $(\pi,\,\mathscr{H}_\pi)$ *be any regular representation of the resolvent algebra* $\mathcal{R}\big(X,\,\sigma\big)$*. For each* $f \in X$

$$j_\pi(f)\pi(R(\lambda,\,f)) = i\lambda\pi(R(\lambda,\,f)) - I \ \textit{on}\ \mathscr{H}_\pi. \tag{2.14}$$

Proof. As noted in [1, Theorem 3.6(vi)] [2, Theorem 4.2(vi)], by using the spectral decomposition of the self-adjoint operator $j_\pi(f)$

$$j_\pi(f)\pi(R(\lambda,\,f)) = \pi(R(\lambda,\,f))j_\pi(f) = i\lambda\pi(R(\lambda,\,f)) - I \quad \text{on } \mathrm{Dom}(j_\pi(f)).$$

As $\pi(R(\lambda,\,f))$ is bounded and $j_\pi(f)$ is a closed operator, this implies (2.14). □

The formula (2.10) can be generalized as follows.

Proposition 2.4. *Let* $(\pi,\,\mathscr{H}_\pi)$ *be an arbitrary regular representation of the resolvent algebra* $\mathcal{R}\big(X,\,\sigma\big)$*. Let* $\xi \in \mathscr{H}_\pi$ *be any vector satisfying that*

$$\xi \in \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_n)\big) \ \textit{ for all } f_1,\cdots,f_n \in X,\ n \in \mathbb{N}. \tag{2.15}$$

That is, $\xi \in \mathscr{H}_\pi$ *is in the common domain for all the polynomials of boson field operators. Then the dense* $*$*-subalgebra* $\mathcal{R}_0\big(X,\,\sigma\big)$ *given in* (2.3) *preserves this property*:

$$\pi\left(\mathcal{R}_0\big(X,\,\sigma\big)\right)\xi \subset \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_n)\big) \ \textit{ for all } f_1,\cdots,f_n \in X,\ n \in \mathbb{N}. \tag{2.16}$$

Proof. Every element of $\mathcal{R}_0\big(X,\,\sigma\big)$ is written as a finite sum of monomials in the form of

$$R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m).$$

Hence for arbitrarily chosen $m \in \mathbb{N}$ and $g_1,\cdots,g_m \in X$, $\mu_1,\cdots,\mu_m \in \mathbb{R}\setminus\{0\}$, it suffices to show that

$$\pi\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\xi \in \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_n)\big) \tag{2.17}$$

for any $n \in \mathbb{N}$ and $f_1,\cdots,f_n \in X$. Due to $\xi \in \mathrm{Dom}\big(j_\pi(f_n)\big)$ by the assumption (2.15), we can use the commutation relation (2.11) repeatedly to obtain $\pi\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\xi \in \mathrm{Dom}\big(j_\pi(f_n)\big)$ and

$$\begin{aligned}
&j_\pi(f_n)\pi\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\xi \\
&= \pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)j_\pi(f_n)\xi \\
&\quad + i\sum_{i=1}^{m}\sigma(f_n,g_i)\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_i,\,g_i)^2\cdots R(\mu_m,\,g_m)\Big)\xi,
\end{aligned} \tag{2.18}$$

where each term '$R(\mu_1,\, g_1)\cdots R(\mu_i,\, g_i)^2\cdots R(\mu_m,\, g_m)$', $i\in\{1,\cdots,m\}$, appeared in the summation formula is understood that we take $R(\mu_i,\, g_i)^2$ for the fixed index i and take $R(\mu_k,\, g_k)$ for the other $k\neq i$.

Next we note that $j_\pi(f_n)\xi\in \mathrm{Dom}\big(j_\pi(f_{n-1})\big)$ and $\xi\in \mathrm{Dom}\big(j_\pi(f_{n-1})\big)$ by the assumption (2.15). Then by using (2.11) repeatedly on the right-hand side of (2.18) as before, we obtain $\pi\Big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\Big)j_\pi(f_n)\xi\in \mathrm{Dom}\big(j_\pi(f_{n-1})\big)$ and $\pi\Big(R(\mu_1,\, g_1)\cdots R(\mu_i,\, g_i)^2\cdots R(\mu_m,\, g_m)\Big)\xi\in \mathrm{Dom}\big(j_\pi(f_{n-1})\big)$ for every $i\in\{1,\cdots,m\}$. These imply $j_\pi(f_n)\pi\Big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\Big)\xi\in \mathrm{Dom}\big(j_\pi(f_{n-1})\big)$. In the same way as done above we have $j_\pi(f_{n-1})j_\pi(f_n)\pi\Big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\Big)\xi\in \mathrm{Dom}\big(j_\pi(f_{n-2})\big)$. So inductively we obtain (2.17). $\square$

Remark 2.5. By using Proposition 2.4 a common invariant domain for all the polynomials of boson field operators can be enlarged to an invariant domain for all the polynomials of resolvents and boson field operators. Proposition 2.4 slightly refines the discussion given below [1, Theorem 3.6].

2.3 Quasi-free dynamics on the resolvent algebra

Let $\mathrm{Sp}\,(X,\sigma)$ denote the symplectic group of X, the group of all bijections T of X which leave the symplectic form σ invariant,

$$\sigma(Tf,Tg)=\sigma(f,g)\quad\text{for all } f,g\in X. \tag{2.19}$$

For any $T\in \mathrm{Sp}\,(X,\sigma)$ there exists a unique $*$-automorphism α_T on $\mathcal{R}\big(X,\,\sigma\big)$ determined by

$$\alpha_T\left(R(\lambda,\, f)\right):=R(\lambda,\, Tf)\quad\text{for all } f\in X \text{ and } \lambda\in\mathbb{R}\setminus\{0\}. \tag{2.20}$$

We call $\alpha_T\in \mathrm{Aut}\mathcal{R}\big(X,\,\sigma\big)$ the Bogoliubov automorphism on $\mathcal{R}\big(X,\,\sigma\big)$ with respect to $T\in \mathrm{Sp}\,(X,\sigma)$, see [1, Theorem 3.4(vi)].

Let $\{T_t;\ t\in\mathbb{R}\}$ be a one parameter group of $\mathrm{Sp}\,(X,\sigma)$. For each $t\in\mathbb{R}$ let $\alpha_t:=\alpha_{T_t}\in \mathrm{Aut}\mathcal{R}\big(X,\,\sigma\big)$. Then $\{\alpha_t\in \mathrm{Aut}\mathcal{R}\big(\mathscr{S}(\mathbb{R},\,\mathbb{R}),\,\sigma\big);\ t\in\mathbb{R}\}$ gives a one-parameter automorphism group of $\mathcal{R}\big(X,\,\sigma\big)$. It is called the quasi-free automorphism group associated with $\{T_t\in \mathrm{Sp}\,(X,\sigma);\ t\in\mathbb{R}\}$. Assume that $\{T_t;\ t\in\mathbb{R}\}$ is differentiable with respect to $t\in\mathbb{R}$. Namely, there exists a linear map S on X such that for every $f\in X$

$$\lim_{t\to 0}\frac{\sigma(T_tf-f,g)}{t}=\sigma\left(S(f),g\right)\quad\text{for all } g\in X. \tag{2.21}$$

3 Extension to the algebra of resolvents and boson field operators

Let us form the abstract algebra

$$\mathcal{E}_{\mathrm{bos}}\big(X,\,\sigma\big):=*\text{-alg}\big\{R(\lambda,\, f),\ j(f)\mid f\in X,\ \lambda\in\mathbb{R}\setminus\{0\}\big\}. \tag{3.1}$$

It is algebraically generated by all the resolvents over $(X,\,\sigma)$ satisfying (2.1) and the *abstract* boson field operators defined as

$$j(f)\equiv i1-R(1,\, f)^{-1}\quad\text{for every } f\in X. \tag{3.2}$$

This implies

$$R(\lambda,\, f)=(i\lambda 1-j(f))^{-1}\quad\text{for every } f\in X\ \text{ and } \lambda\in\mathbb{R}\setminus\{0\}. \tag{3.3}$$

From (3.2) (2.1) we derive the canonical commutation relations for $\{j(f);\ f \in X\}$:

$$j(f)^* = j(f), \quad [j(f),\ j(g)] = i\sigma(f,g)1 \ \text{ for all } \ f, g \in X. \tag{3.4}$$

We shall embed the dense subalgebra $\mathcal{R}_0\big(X,\,\sigma\big)$ in the C^*-algebra $\mathcal{R}\big(X,\,\sigma\big)$ into $\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)$ as a $*$-subalgebra. The following proposition has been noted below [1, Theorem 3.6(i)]. For completeness we will provide its proof.

Proposition 3.1. *Let ω be an arbitrary regular state of $\mathcal{R}\big(X,\,\sigma\big)$ and let $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ denote its GNS representation. Assume that*

$$\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\big) \ \textit{ for all } f_1, \cdots, f_n \in X,\ n \in \mathbb{N}. \tag{3.5}$$

Then the state ω on $\mathcal{R}\big(X,\,\sigma\big)$ can be extended to the $*$-algebra $\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)$ by determining

$$\omega\left(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)j(f_1)\cdots j(f_n)\right) := \Big(\Omega_\omega,\ \pi_\omega\left(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\right) j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\Omega_\omega\Big) \tag{3.6}$$

for all $f_1, \cdots, f_n,\, g_1, \cdots, g_m \in X,\ \mu_1, \cdots, \mu_m \in \mathbb{R}\setminus\{0\},\ n, m \in \mathbb{N}$. Similarly the GNS representation $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ extends to $\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)$ by setting

$$\pi_\omega\big(j(f)\big) := j_{\pi_\omega}(f) \ \textit{ for every } f \in X.$$

Proof. Take

$$\pi_\omega\left(\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)\right) := *\text{-alg}\big\{j_{\pi_\omega}(f),\ \pi_\omega\left(R(\lambda,\,f)\right) \mid f \in X,\ \lambda \in \mathbb{R}\setminus\{0\}\big\}.$$

We will clarify this *formal* algebra as an algebra of linear operators on $\mathscr{H}_\omega$. As shown in Proposition 2.4, every element of $\pi_\omega\left(\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)\right)$ can be written as a finite sum of monomials in the form

$$\pi_\omega\left(R(\mu_1,\,g_1)\right)\cdots \pi_\omega\left(R(\mu_m,\,g_m)\right) j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)$$

on the common domain for all the polynomials of boson field operators. By the assumption (3.5) the cyclic vector Ω_ω is in this common domain and the right-hand side of (3.6) is meaningful. Therefore the expression (3.6) determines uniquely expectation values with respect to ω for all the operators in $\pi_\omega\left(\mathcal{E}_{\rm bos}\big(X,\,\sigma\big)\right)$. ☐

4 Analytic properties of quasi-free dynamics on the resolvent algebra

We characterize the quasi-free dynamics and the regular states which we will consider.

Definition 4.1. Suppose that $\{T_t \in \mathrm{Sp}\,(X, \sigma)\,;\ t \in \mathbb{R}\}$ is a differentiable one parameter group of $\mathrm{Sp}\,(X, \sigma)$ and that S denotes its infinitesimal generator on X as defined in (2.21). Let $\{\alpha_t \in \mathrm{Aut}\mathcal{R}\big(X,\,\sigma\big)\,;\ t \in \mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{R}\big(X,\,\sigma\big)$ associated with $\{T_t \in \mathrm{Sp}\,(X, \sigma)\,;\ t \in \mathbb{R}\}$. Let ω be a regular state on $\mathcal{R}\big(X,\,\sigma\big)$ and let $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ denote its GNS representation. Assume that

$$\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\big) \ \text{ for all } f_1, \cdots, f_n \in X,\ n \in \mathbb{N}, \tag{4.1}$$

and that

$$\lim_{t\to 0} \frac{j_{\pi_\omega}(T_t f) - j_{\pi_\omega}(f)}{t}\Omega_\omega = j_{\pi_\omega}\left(S(f)\right)\Omega_\omega \ \text{ for every } f \in X, \tag{4.2}$$

where the limit on the left-hand side is taken with respect to the strong topology of $\mathscr{H}_\omega$. For the given $\{\alpha_t \in \mathrm{Aut}\mathcal{R}\big(X,\,\sigma\big)\,;\ t \in \mathbb{R}\}$ the set of all regular states on $\mathcal{R}\big(X,\,\sigma\big)$ satisfying (4.1) and (4.2) is denoted by $K_{\alpha_t}\big(\mathcal{R}\big(X,\,\sigma\big)\big)$.

Remark 4.2. Both (4.1) (4.2) in Definition 4.1 are natural requirements of quantum field theory. The first one (4.1) is tightly related to 'invariant domains for fields' in the Gårding-Wightman axioms [4, Property 4, IX.8], and it has already appeared as (2.15) in Proposition 2.4. The second one (4.2) is tightly related to 'regularity of the field' in the Gårding-Wightman axioms [4, Property 5, IX.8], and it will be referred to as j-field differentiability with respect to $\{\alpha_t \in \mathrm{Aut}\mathcal{R}(X,\,\sigma)\,;\, t\in\mathbb{R}\}$.

The infinitesimal formula (4.2) of Definition 4.1 can be extended with no extra input as follows.

Lemma 4.3. *Let $\omega \in K_{\alpha_t}(\mathcal{R}(X,\,\sigma))$ of Definition 4.1, namely ω is an arbitrary regular state of $\mathcal{R}(X,\,\sigma)$ that satisfies the conditions (4.1) and (4.2) with respect to the quasi-free automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{R}(X,\,\sigma)\,;\, t\in\mathbb{R}\}$ associated with $\{T_t \in \mathrm{Sp}\,(X,\sigma)\,;\, t\in\mathbb{R}\}$. Then for every $f\in X$,*

$$\frac{d}{dt} j_{\pi_\omega}(T_t f)\xi\,\Big|_{t=0} = j_{\pi_\omega}(S(f))\xi \quad \textit{for each}\ \ \xi \in \pi_\omega\left(\mathcal{R}_0(X,\,\sigma)\right)\Omega_\omega, \tag{4.3}$$

where $\mathcal{R}_0(X,\,\sigma)$ is the dense $*$-subalgebra of $\mathcal{R}(X,\,\sigma)$ given in (2.3), and the derivation on the left-hand side is taken with respect to the strong topology of $\mathscr{H}_\omega$.

Proof. As every $\xi \in \pi_\omega(\mathcal{R}_0(X,\,\sigma))\Omega_\omega$ is written by a finite sum of vectors of the form $\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega$, with $g_1,\cdots,g_m \in X$, $\mu_1,\cdots,\mu_m \in \mathbb{R}\setminus\{0\}$ and $m\in\mathbb{N}$, it suffices to show (4.3) for such vectors. Note that $\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(T_t f)\big)$ for all $t\in\mathbb{R}$ due to (4.1). As (2.18) in the proof of Proposition 2.4 we have

$$\begin{aligned}
& j_{\pi_\omega}(T_t f)\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega \\
&= \pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big) j_{\pi_\omega}(T_t f)\Omega_\omega \\
&\quad + i\sum_{i=1}^{m}\sigma(T_t f, g_i)\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_i,\,g_i)^2\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega.
\end{aligned} \tag{4.4}$$

By this formula together with (2.21) (4.2) we have

$$\begin{aligned}
& \lim_{t\to 0}\frac{j_{\pi_\omega}(T_t f) - j_{\pi_\omega}(f)}{t}\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega \\
&= \pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big) j_{\pi_\omega}(S(f))\Omega_\omega \\
&\quad + i\sum_{i=1}^{m}\sigma(S(f), g_i)\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_i,\,g_i)^2\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega \\
&= j_{\pi_\omega}(S(f))\pi_\omega\Big(R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\Big)\Omega_\omega.
\end{aligned} \tag{4.5}$$

We have shown the statement. ☐

We will show the main result in this section. (No analogous result is given in [1, 2].)

Proposition 4.4. *Suppose that $\{T_t \in \mathrm{Sp}\,(X,\sigma)\,;\, t\in\mathbb{R}\}$ is a differentiable one parameter group of $\mathrm{Sp}\,(X,\sigma)$. Let S denote its infinitesimal generator defined on X. Let $\{\alpha_t \in \mathrm{Aut}\mathcal{R}(X,\,\sigma)\,;\, t\in\mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{R}(X,\,\sigma)$ associated with $\{T_t \in \mathrm{Sp}\,(X,\sigma)\,;\, t\in\mathbb{R}\}$. Let $\omega \in K_{\alpha_t}(\mathcal{R}(X,\,\sigma))$ of Definition 4.1. For every $f\in X$ and $\lambda\in\mathbb{R}\setminus\{0\}$*

$$\frac{d}{dt}\pi_\omega\left(\alpha_t\left(R(\lambda,\,f)\right)\right)\xi\Big|_{t=0} = \pi_\omega(R(\lambda,\,f))j_{\pi_\omega}\left(S(f)\right)\pi_\omega(R(\lambda,\,f))\xi \quad \textit{for each}\ \ \xi\in\pi_\omega\left(\mathcal{R}_0(X,\,\sigma)\right)\Omega_\omega, \tag{4.6}$$

where the derivation on the left-hand side is taken with respect to the strong topology of $\mathscr{H}_\omega$.

Proof. For each $t \in \mathbb{R}$

$$\begin{aligned}\pi_\omega\left(\alpha_t\big(R(\lambda,\, f)\big) - R(\lambda,\, f)\right)\xi &= \pi_\omega\left(R(\lambda,\, T_t f) - R(\lambda,\, f)\right)\xi \\ &= \pi_\omega\left(R(\lambda,\, T_t f)\right)\Big(\pi_\omega\left(R(\lambda,\, f)\right)^{-1} - \pi_\omega\left(R(\lambda,\, T_t f)\right)^{-1}\Big)\pi_\omega\left(R(\lambda,\, f)\right)\xi.\end{aligned}$$

As $\pi_\omega(R(\lambda,\, f))\xi \in \pi_\omega\left(\mathcal{R}_0\big(X,\, \sigma\big)\right)\Omega_\omega$, by the assumption (4.1) we can use Proposition 2.4 to see $\pi_\omega(R(\lambda,\, f))\xi \in \mathrm{Dom}\big(j_{\pi_\omega}(T_t f)\big)$ for every $t \in \mathbb{R}$. By substituting (2.6) to the above equation we obtain

$$\pi_\omega\left(\alpha_t\big(R(\lambda,\, f)\big) - R(\lambda,\, f)\right)\xi = \pi_\omega\left(R(\lambda,\, T_t f)\right)\big(j_{\pi_\omega}(T_t f) - j_{\pi_\omega}(f)\big)\pi_\omega(R(\lambda,\, f))\xi.$$

As $\pi_\omega(R(\lambda,\, f))\xi \in \pi_\omega\left(\mathcal{R}_0\big(X,\, \sigma\big)\right)\Omega_\omega$, from (4.3) in Lemma 4.3 it follows that

$$\lim_{t\to 0}\frac{1}{t}\pi_\omega\left(\alpha_t\big(R(\lambda,\, f)\big) - R(\lambda,\, f)\right)\xi = \pi_\omega\left(R(\lambda,\, f)\right)\big(j_{\pi_\omega}\left(S(f)\right)\big)\pi_\omega(R(\lambda,\, f))\xi.$$

This is the desired formula (4.6). □

5 Supersymmetric dynamics

We shall focus on the supersymmetric C^*-dynamics given in [1]. This is a simplified supersymmetry model made by a spinless neutral fermion field and a neutral boson field on one-dimensional space-time $\mathbb{R}$ (which may be interpreted as the forwarded light-ray in the two dimensional Minkowski space-time). The underlying function space for these quantum fields is given by $\mathscr{S}(\mathbb{R},\, \mathbb{R})$, i.e. the $\mathbb{R}$-valued Schwartz space on $\mathbb{R}$.

5.1 Fermion-boson C^*-system

Let us define the fermion-boson C^*-system. We take the Clifford operators satisfying that

$$c(f) = c(f)^*, \quad \{c(f), c(g)\} = \tau(f, g)1 \ \text{ for all } \ f, g \in \mathscr{S}(\mathbb{R},\, \mathbb{R}), \tag{5.1}$$

where the real scalar product τ on $\mathscr{S}(\mathbb{R},\, \mathbb{R})$ is given as

$$\tau(f, g) := \int_{\mathbb{R}} f(x)g(x)dx \ \text{ for } \ f, g \in \mathscr{S}(\mathbb{R},\, \mathbb{R}). \tag{5.2}$$

Let $\mathrm{Cliff}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\big) \equiv C^*\{c(f) \mid f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})\}$ which is called the Clifford algebra over $\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\big)$ as usual. This unital C^*-algebra gives the fermion system. The group of all the orthogonal operators on $(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau)$ is denoted by $\mathcal{O}\left(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\right)$.

For the boson system we take the resolvent algebra $\mathcal{R}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big)$ over $(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma)$ as in Sect.2.1, where the symplectic form is given by

$$\sigma(f, g) := \int_{\mathbb{R}} f(x)g'(x)dx \ \text{ for } \ f, g \in \mathscr{S}(\mathbb{R},\, \mathbb{R}). \tag{5.3}$$

The symplectic group on $(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma)$ is denoted by $\mathrm{Sp}\left(\mathscr{S}(\mathbb{R},\, \mathbb{R}), \sigma\right)$.

The total fermion-boson system is given by the unique C^* tensor-product of the Clifford algebra and the resolvent algebra:

$$\mathcal{F} := \mathrm{Cliff}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\big) \otimes \mathcal{R}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big). \tag{5.4}$$

Both $\mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$ and $\mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$ are imbedded into $\mathcal{F}$ as subalgebras. The grading automorphism γ on $\mathcal{F}$ is determined by

$$\gamma(c(f)) = -c(f) \text{ for all } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}). \tag{5.5}$$

Precisely, $\gamma = \gamma_1 \otimes i_2$, where γ_1 is the grading automorphism on $\mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$ satisfying that $\gamma_1(c(f)) = -c(f)$ for all $f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$, and i_2 is the identity on $\mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$. The fermion-boson system $\mathcal{F}$ is decomposed into the even part $\mathcal{F}_+$ and the odd part $\mathcal{F}_-$ by the grading automorphism γ as follows:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \quad \mathcal{F}_+ := \{F \in \mathcal{F} \mid \gamma(F) = F\}, \quad \mathcal{F}_- := \{F \in \mathcal{F} \mid \gamma(F) = -F\}. \tag{5.6}$$

As defined previously in (2.3), $\mathcal{R}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$ denotes the polynomial $*$-subalgebra generated by all the resolvent elements of $\mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$:

$$\mathcal{R}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma) \equiv *\text{-alg}\{R(\lambda, f) \mid f \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R} \setminus \{0\}\} \subset \mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma). \tag{5.7}$$

Similarly let us introduce

$$\mathrm{Cliff}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau) := *\text{-alg}\{c(f) \mid f \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R} \setminus \{0\}\} \subset \mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau), \tag{5.8}$$

and

$$\mathcal{F}_0 := *\text{-alg}\{c(f),\ R(\lambda, f) \mid f \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R} \setminus \{0\}\} \subset \mathcal{F}. \tag{5.9}$$

By definition $\mathcal{R}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$ is norm dense in $\mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$, and also $\mathrm{Cliff}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$ is norm dense in $\mathrm{Cliff}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$. Hence $\mathcal{F}_0$ is a unital γ-invariant norm dense $*$-subalgebra of $\mathcal{F}$.

5.2 Regular representations and regular states on the fermion-boson C^*-system

The definition of regular representations given in Sect.2.2 makes sense for the fermion-boson C^*-system $\mathcal{F}$ into which the resolvent algebra is imbedded as in (5.4). Let π be a $*$-representation of $\mathcal{F}$ upon a Hilbert space $\mathscr{H}_\pi$. If $(\pi,\ \mathscr{H}_\pi)$ satisfies (2.4), then it is called regular. Let ω be a state of $\mathcal{F}$. If its GNS representation is regular, then ω is called regular. For any regular representation $(\pi,\ \mathscr{H}_\pi)$ of $\mathcal{F}$, boson field operators $j_\pi(f)$ ($f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$) are given as (2.5). It is easy to see that each $j_\pi(f)$ is a self-adjoint operator on $\mathscr{H}_\pi$. Likewise other basic properties of regular representations listed in Sect.2.2 are valid for the fermion-boson system $\mathcal{F}$. The exactly same statement as Proposition 2.4 holds for $\mathcal{F}$ in place of $\mathcal{R}(X, \sigma)$. We will show a generalization of Proposition 2.4 that incorporates fermion field operators.

Proposition 5.1. *Let $(\pi,\ \mathscr{H}_\pi)$ be an arbitrary regular representation of the fermion-boson C^*-system $\mathcal{F}$. Let $\xi \in \mathscr{H}_\pi$ be any vector satisfying that*

$$\xi \in \mathrm{Dom}(j_\pi(f_1) \cdots j_\pi(f_n)) \quad \textit{for all } f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}. \tag{5.10}$$

Then the dense $*$-subalgebra $\mathcal{F}_0$ given in (5.9) *preserves this property:*

$$\pi(\mathcal{F}_0)\, \xi \subset \mathrm{Dom}(j_\pi(f_1) \cdots j_\pi(f_n)) \quad \textit{for all } f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}. \tag{5.11}$$

Proof. Every element of $\mathcal{F}_0$ is written as a finite sum of monomials in the form of

$$c(h_1) \cdots c(h_k) R(\mu_1,\, g_1) \cdots R(\mu_m,\, g_m).$$

Take arbitrarily $k, m \in \mathbb{N}$ and $h_1, \cdots, h_k, g_1, \cdots, g_m \in \mathscr{S}(\mathbb{R}, \mathbb{R})$, $\mu_1, \cdots, \mu_m \in \mathbb{R} \setminus \{0\}$. Then it suffices to show that

$$\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \in \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_n)\big) \tag{5.12}$$

for any $n \in \mathbb{N}$ and $f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R})$. In the following we fix $n \in \mathbb{N}$ and $f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R})$.

As a first step, let us verify

$$\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \in \mathrm{Dom}\big(j_\pi(f_n)\big) \tag{5.13}$$

and

$$j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi = \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_n)\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi. \tag{5.14}$$

By the assumption (5.10), it follows from Proposition 2.4 (which holds obviously for the present setting as noted) that $\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \in \mathscr{H}_\pi$ is in the domain for all the polynomials of boson field operators, especially in $\mathrm{Dom}\big(j_\pi(f_n)\big)$. By $\mathrm{Dom}\big(j_\pi(f_n)\big) = \mathrm{Ran}\left(\pi\big(R(1, f_n)\big)\right)$ as noted in (2.5), there exists an $\eta \in \mathscr{H}_\pi$ such that

$$\pi\left(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\right)\xi = \pi(R(1, f_n))\eta. \tag{5.15}$$

By (5.15) and the commutativity between (fermion) Clifford operators and (boson) resolvent operators,

$$\begin{aligned}
\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi &= \pi\big(c(h_1)\cdots c(h_k)\big)\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)\pi(R(1, f_n))\eta \\
&= \pi(R(1, f_n))\pi\big(c(h_1)\cdots c(h_k)\big)\eta.
\end{aligned} \tag{5.16}$$

By noting $\mathrm{Dom}\big(j_\pi(f_n)\big) = \mathrm{Ran}\left(\pi\big(R(1, f_n)\big)\right)$ again this yields (5.13). We can derive (5.14) from the above (5.15) (5.16) together with the identity (2.14) of Lemma 2.3 as

$$\begin{aligned}
&j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \\
&= j_\pi(f_n)\pi(R(1, f_n))\pi\big(c(h_1)\cdots c(h_k)\big)\eta \\
&= \left(i\pi(R(1, f_n)) - I\right)\pi\big(c(h_1)\cdots c(h_k)\big)\eta \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)\left(i\pi(R(1, f_n)) - I\right)\eta \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_n)\pi(R(1, f_n))\eta \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_n)\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi.
\end{aligned}$$

Next let us show

$$\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \in \mathrm{Dom}\big(j_\pi(f_{n-1})j_\pi(f_n)\big) \tag{5.17}$$

and

$$\begin{aligned}
&j_\pi(f_{n-1})j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_{n-1})j_\pi(f_n)\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi.
\end{aligned} \tag{5.18}$$

It follows from Proposition 2.4 that $\pi\big(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\big)\xi$ is in the domain for all the polynomials of boson field operators, especially in $\mathrm{Dom}\big(j_\pi(f_{n-1})j_\pi(f_n)\big)$. By $\mathrm{Dom}\big(j_\pi(f_{n-1})\big) = \mathrm{Ran}\left(\pi\big(R(1, f_{n-1})\big)\right)$ there exists an $\eta' \in \mathscr{H}_\pi$ such that

$$j_\pi(f_n)\pi\left(R(\mu_1, g_1)\cdots R(\mu_m, g_m)\right)\xi = \pi(R(1, f_{n-1}))\eta'. \tag{5.19}$$

By (5.19) (5.14) we can verify (5.17) as

$$\begin{aligned}
j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi &= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)\pi(R(1,\, f_{n-1}))\eta' \\
&= \pi(R(1,\, f_{n-1}))\pi\big(c(h_1)\cdots c(h_k)\big)\eta' \in \mathrm{Dom}\big(j_\pi(f_{n-1})\big).
\end{aligned} \tag{5.20}$$

By using (5.19) (5.20) we verify (5.18) as

$$\begin{aligned}
&j_\pi(f_{n-1})j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi \\
&= j_\pi(f_{n-1})\pi(R(1,\, f_{n-1}))\pi\big(c(h_1)\cdots c(h_k)\big)\eta' \\
&= (i\pi(R(1,\, f_{n-1})) - I)\,\pi\big(c(h_1)\cdots c(h_k)\big)\eta' \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)\,(i\pi(R(1,\, f_{n-1})) - I)\,\eta' \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_{n-1})\pi(R(1,\, f_{n-1}))\eta' \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_{n-1})j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi.
\end{aligned}$$

Repeating the above procedure from $j_\pi(f_n)$ down to $j_\pi(f_1)$, we can show (5.12) and

$$\begin{aligned}
&j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(c(h_1)\cdots c(h_k)R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi \\
&= \pi\big(c(h_1)\cdots c(h_k)\big)j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m)\big)\xi.
\end{aligned} \tag{5.21}$$

Thus we have completed the proof. □

Remark 5.2. By Proposition 5.1 a common invariant domain for all the polynomials of boson field operators is enlarged to an invariant domain for all the polynomials of Clifford operators, resolvent operators and boson field operators.

Remark 5.3. The regular representation (or the regular state) in Proposition 5.1 is not necessarily the product type between $\mathrm{Cliff}\big(\mathscr{S}(\mathbb{R},\,\mathbb{R}),\,\tau\big)$ and $\mathcal{R}\big(\mathscr{S}(\mathbb{R},\,\mathbb{R}),\,\sigma\big)$. No matter how it has strong correlation between the fermion system and the boson system, no fermion field operator can break the common domain for all the polynomials of boson fields.

Remark 5.4. Proposition 5.1 is given for the particular fermion-boson C^*-system $\mathcal{F} = \mathrm{Cliff}\big(\mathscr{S}(\mathbb{R},\,\mathbb{R}),\,\tau\big)\otimes\mathcal{R}\big(\mathscr{S}(\mathbb{R},\,\mathbb{R}),\,\sigma\big)$ defined in Sect.5.1. It is obvious that the same statement holds for a general space X (in place of $\mathscr{S}(\mathbb{R},\,\mathbb{R})$) with any real scalar product τ and any non-degenerate symplectic form σ on it.

We can see that fermion field operators commute with boson field operators as well as resolvent operators. We have to take care of the domain on which these operators act.

Corollary 5.5. *Let $(\pi,\ \mathscr{H}_\pi)$ be an arbitrary regular representation of the fermion-boson C^*-system $\mathcal{F}$. Then any polynomial generated by fermion field operators and any polynomial generated by boson field operators and resolvent operators commute with each other on any vector $\xi \in \mathscr{H}_\pi$ such that*

$$\xi \in \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_l)\big) \ \ \textit{for all } f_1,\cdots,f_l \in \mathscr{S}(\mathbb{R},\,\mathbb{R}),\ l\in\mathbb{N}.$$

Proof. From (5.21) in the proof Proposition 5.1 it follows that for any $k,n,m \in \mathbb{N}$ and any $h_1,\cdots,h_k,\, f_1,\cdots,f_n,\, g_1,\cdots,g_m \in \mathscr{S}(\mathbb{R},\,\mathbb{R}),\, \mu_1,\cdots,\mu_m \in \mathbb{R}\setminus\{0\}$

$$\begin{aligned}
&\pi\big(c(h_1)\big)\cdots\pi\big(c(h_k)\big)j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\big)\cdots\pi\big(R(\mu_m,\, g_m)\big)\xi \\
&= j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(c(h_1)\big)\cdots\pi\big(c(h_k)\big)\pi\big(R(\mu_1,\, g_1)\big)\cdots\pi\big(R(\mu_m,\, g_m)\big)\xi \\
&= j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\big)\cdots\pi\big(R(\mu_m,\, g_m)\big)\pi\big(c(h_1)\big)\cdots\pi\big(c(h_k)\big)\xi.
\end{aligned} \tag{5.22}$$

The above fermion-boson commutativity (5.22) implies the assertion fully, since as shown in (the proof of) Proposition 5.1 every polynomial generated by boson field operators and resolvent operators is given as a finite sum of monomials in the form $j_\pi(f_1)\cdots j_\pi(f_n)\pi\big(R(\mu_1,\, g_1)\big)\cdots\pi\big(R(\mu_m,\, g_m)\big)$ on $\pi(\mathcal{F}_0)\xi$. □

5.3 Defining a superderivation and a time automorphism group on the fermion-boson C^*-system

As in [1, Sect.4] we set

$$\zeta(f) := c(f)R(1,\, f) \in \mathcal{F}_- \text{ for each } f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}). \tag{5.23}$$

These $\zeta(f) \in \mathcal{F}_-$ ($f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$) will be called *mollified fermion field operators*. For every $f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$

$$\zeta(f)^* = R(1,\, f)^* c(f)^* = R(-1,\, f)c(f) = c(f)R(-1,\, f) = c(f)\frac{1}{-1}R(1,\, \tfrac{1}{-1}f) = c(-f)R(1,\, -f) = \zeta(-f). \tag{5.24}$$

Note that

$$\lambda\zeta(\tfrac{1}{\lambda}f) = \lambda c(\tfrac{f}{\lambda})R\big(1,\, \tfrac{f}{\lambda}\big) = c(f)R\big(1,\, \tfrac{f}{\lambda}\big) \text{ for every } f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}) \text{ and } \lambda \in \mathbb{R}\setminus\{0\}. \tag{5.25}$$

By using (5.4) (2.2) (5.1) we have

$$\|\lambda\zeta(\tfrac{1}{\lambda}f)\| = \|c(f)\| = \sqrt{\tfrac{\tau(f,f)}{2}} \text{ for every } f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}) \text{ and } \lambda \in \mathbb{R}\setminus\{0\}. \tag{5.26}$$

For any regular representation $\big(\pi,\, \mathscr{H}_\pi\big)$ of $\mathcal{F}$, it follows from (2.13) (5.25) that for each $f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$

$$\pi\left(c(f)\right) = \lim_{\lambda\to\infty} i\pi\left(\lambda\zeta(\tfrac{1}{\lambda}f)\right) \text{ in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi). \tag{5.27}$$

By this one can asymptotically recover a fermion field operator from its mollified ones.

We shall formulate a superderivation on the graded C^*-algebra $\mathcal{F}$ making use of mollified fermion field operators. We take the unital $*$-subalgebra generated by all the mollified fermion field operators and all the resolvents:

$$\mathcal{A}_\circ := *\text{-alg}\big\{\zeta(f),\ R(\lambda,\, f) \mid f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}),\ \lambda \in \mathbb{R}\setminus\{0\}\big\} \subset \mathcal{F}_0. \tag{5.28}$$

It is easy to see that $\mathcal{A}_\circ$ is γ-invariant. For every $f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$ and $\lambda \in \mathbb{R}\setminus\{0\}$ set

$$\begin{aligned} \delta_{\mathrm{s}}(\zeta(f)) &:= iR(1,\, f) - 1 - ic(f)c(f')R(1,\, f)^2 \in \mathcal{F}_0, \\ \delta_{\mathrm{s}}(R(\lambda,\, f)) &:= ic(f')R(\lambda,\, f)^2 \in \mathcal{F}_0. \end{aligned} \tag{5.29}$$

These uniquely determine a superderivation $\delta_{\mathrm{s}} : \mathcal{A}_\circ \to \mathcal{F}_0$. See [1, Sect.4] for the detail. The superderivation δ_{s} is hermite, as its conjugation

$$\delta_{\mathrm{s}}^*(A) := -\big(\delta_{\mathrm{s}}\big(\gamma(A^*)\big)\big)^* \quad \text{for every } \ A \in \mathcal{A}_\circ, \tag{5.30}$$

is equal to δ_{s}.

For every $A \in \mathcal{A}_\circ$, there exists a net $\big\{M_{A,\lambda} \in \mathcal{R}_0\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big) \,\big|\, \lambda \in \mathbb{R}_+\big\}$ such that

$$M_{A,\lambda}\delta_{\mathrm{s}}(A) \in \mathcal{A}_\circ \text{ for every } \ \lambda \in \mathbb{R}_+, \tag{5.31}$$

and that for any regular representation $\big(\pi,\, \mathscr{H}_\pi\big)$ of $\mathcal{F}$,

$$\pi(M_{A,\lambda}) \longrightarrow I \quad \text{in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi) \ \text{ as } \ \lambda\to\infty, \tag{5.32}$$

and also

$$\pi\big(\delta_{\mathrm{s}}(M_{A,\lambda})\big) \longrightarrow 0 \quad \text{in the strong operator topology of } \ \mathfrak{B}(\mathscr{H}_\pi) \ \text{ as } \ \lambda\to\infty. \tag{5.33}$$

Such $\{M_{A,\lambda} \in \mathcal{R}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma) \mid \lambda \in \mathbb{R}_+\}$ is called the set of *mollifiers with respect to the superderivation* δ_{s}. The explicit formula of $M_{A,\lambda}$ is given in [1] [3]. For each $A \in \mathcal{A}_\circ$ set

$$M_{A,\lambda}\delta_{\mathrm{s}}^2(A) := \delta_{\mathrm{s}}\left(M_{A,\lambda}\delta_{\mathrm{s}}(A)\right) - \delta_{\mathrm{s}}(M_{A,\lambda})\delta_{\mathrm{s}}(A) \in \mathcal{F}_0 \quad \text{for} \quad \lambda \in \mathbb{R}_+. \tag{5.34}$$

Note that $M_{A,\lambda}\delta_{\mathrm{s}}^2(A)$ is not the product of $M_{A,\lambda}$ and $\delta_{\mathrm{s}}^2(A)$; '$\delta_{\mathrm{s}}^2(A)$' does not necessarily denote an element of $\mathcal{F}$, rather it is a heuristic symbol.

We will formulate a time automorphism group on the graded C^*-algebra $\mathcal{F}$. For a given $f(x) \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ let $f_t(x) \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ denote its translation by $t \in \mathbb{R}$:

$$f_t(x) \equiv f(x-t). \tag{5.35}$$

For each $t \in \mathbb{R}$ let T_t denote the bijection on $\mathscr{S}(\mathbb{R}, \mathbb{R})$ given as

$$T_t f := f_t \quad \text{for all } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}). \tag{5.36}$$

Its derivation is given by the differential map on $\mathscr{S}(\mathbb{R}, \mathbb{R})$,

$$S(f) := -f' \quad \text{for all } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}). \tag{5.37}$$

It is easy to see that $T_t \in \mathcal{O}\left(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau\right)$ and $T_t \in \mathrm{Sp}\left(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma\right)$ for every $t \in \mathbb{R}$. We provide a quasi-free automorphism group $\{\alpha_t\,;\ t \in \mathbb{R}\}$ of $\mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$ associated with $\{T_t \in \mathcal{O}\left(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau\right);\ t \in \mathbb{R}\}$ by

$$\alpha_t(c(f)) := c(f_t) \quad \text{for all } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}) \tag{5.38}$$

and similarly a quasi-free automorphism group $\{\alpha_t\,;\ t \in \mathbb{R}\}$ of $\mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma)$ associated with $\{T_t \in \mathrm{Sp}\left(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma\right);\ t \in \mathbb{R}\}$ by

$$\alpha_t(R(\lambda,\, f)) := R(\lambda,\, f_t) \quad \text{for all } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R} \setminus \{0\}. \tag{5.39}$$

For each $t \in \mathbb{R}$, define $\alpha_t \in \mathrm{Aut}\mathcal{F}$ by

$$\alpha_t(F_1 \otimes F_2) = \alpha_t(F_1) \otimes \alpha_t(F_2) \quad \text{for every } F_1 \in \mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau) \text{ and } F_2 \in \mathcal{R}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma). \tag{5.40}$$

The one-parameter group of $*$-automorphisms $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ given above will be called the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations by its constriction. This automorphism group has the strong continuity on the fermion system.

Lemma 5.6. *The quasi-free automorphism group associated with the shift translations* $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ *is pointwise norm continuous with respect to* $t \in \mathbb{R}$ *on* $\mathrm{Cliff}(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$. *It is pointwise norm differentiable with respect to* $t \in \mathbb{R}$ *on the subalgebra* $\mathrm{Cliff}_0(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau)$ *given in* (5.8) *satisfying that*

$$\frac{d}{dt}\alpha_t\left(c(f)\right)\Big|_{t=0} = -c(f') \quad \textit{for each} \quad f \in \mathscr{S}(\mathbb{R}, \mathbb{R}). \tag{5.41}$$

Proof. From (5.1) (5.38) (5.40) the statement is obvious. □

Remark 5.7. The infinitesimal generator of the automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ is available on the fermion system as seen in Lemma 5.6, however, it can not be appropriately defined on $\mathcal{F}$, since the automorphism group is not strongly continuous on the boson system.

Note that the domain $\mathcal{A}_\circ$ of δ_{s} is not norm dense in $\mathcal{F}$. However, $\mathcal{A}_\circ$ is dense in $\mathcal{F}$ in a weaker sense.

Lemma 5.8. *Let $(\pi,\ \mathscr{H}_\pi)$ be any regular representation of the fermion-boson C^*-system $\mathcal{F}$. Then $\pi(\mathcal{A}_\circ)$ is dense in $\pi(\mathcal{F})$ in the strong operator topology of $\mathfrak{B}(\mathscr{H}_\pi)$. Let ω be any regular state of $\mathcal{F}$ and let $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ denote its GNS representation. Then*

$$\mathscr{H}_{\omega\,\circ} := \pi_\omega(\mathcal{A}_\circ)\Omega_\omega \tag{5.42}$$

is dense in $\mathscr{H}_\omega$.

Proof. As $\mathcal{F}_0$ is norm dense in $\mathcal{F}$, $\pi(\mathcal{F}_0)$ is norm dense in $\pi(\mathcal{F})$. Hence it suffices to show that for each $B \in \mathcal{F}_0$ one can construct a net in $\pi(\mathcal{A}_\circ)$ which converges to $\pi(B)$ in the strong operator topology of $\mathfrak{B}(\mathscr{H}_\pi)$.

First let us consider $c(f) \in \mathcal{F}_0$ with an arbitrary $f \in \mathscr{S}(\mathbb{R},\mathbb{R})$. (Note that $c(f) \notin \mathcal{A}_\circ$.) A desired convergent net is given by $\big\{i\lambda\zeta(\frac{1}{\lambda}f) \in \mathcal{A}_\circ;\ \lambda \in \mathbb{R}\setminus\{0\}\big\}$, since by (5.27)

$$\pi\left(c(f)\right) = \lim_{\lambda\to\infty} \pi\left(i\lambda\zeta(\tfrac{1}{\lambda}f)\right) \ \text{in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi). \tag{5.43}$$

For later argument we note (5.26),

$$\|\lambda\zeta(\tfrac{1}{\lambda}f)\| = \sqrt{\tfrac{\tau(f,f)}{2}}. \tag{5.44}$$

Next we consider a monomial of $\mathcal{F}_0$. Let

$$B_1 := c(f_1)\cdots c(f_n)R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m) \in \mathcal{F}_0, \tag{5.45}$$

where $n,m \in \mathbb{N}$, $f_1,\cdots,f_n \in \mathscr{S}(\mathbb{R},\mathbb{R})$, $g_1,\cdots,g_m \in \mathscr{S}(\mathbb{R},\mathbb{R})$, $\mu_1,\cdots,\mu_m \in \mathbb{R}\setminus\{0\}$ are arbitrary. For such $B_1 \in \mathcal{F}_0$ we take the net $\{B_1(\lambda) \in \mathcal{A}_\circ;\ \lambda \in \mathbb{R}\setminus\{0\}\}$, where

$$B_1(\lambda) := i\lambda\zeta(\tfrac{f_1}{\lambda})\cdots i\lambda\zeta(\tfrac{f_n}{\lambda})R(\mu_1,\, g_1)\cdots R(\mu_m,\, g_m) \in \mathcal{A}_\circ \quad \text{for each } \lambda \in \mathbb{R}\setminus\{0\}. \tag{5.46}$$

As the multiplication of linear operators on a Hilbert space is jointly continuous in the strong operator topology provided that these operators are uniformly bounded, by (5.43) (5.44) (5.45) (5.46) we have

$$\pi\left(B_1\right) = \lim_{\lambda\to\infty} \pi\left(B_1(\lambda)\right) \ \text{in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi). \tag{5.47}$$

Finally we take an arbitrary element $B \in \mathcal{F}_0$. It is by definition given as a finite sum $\sum_{i=1}^k c_iB_i$, where $c_i \in \mathbb{C}$, $k \in \mathbb{N}$, and each B_i is a monomial of $\mathcal{F}_0$ as given in (5.45). For each monomial $B_i \in \mathcal{F}_0$, $i \in \{1,2,\cdots,k\}$, take $B_i(\lambda) \in \mathcal{A}_\circ$ for every $\lambda \in \mathbb{R}\setminus\{0\}$ in the same way as (5.46). Let

$$B(\lambda) := \sum_{i=1}^{k} c_iB_i(\lambda) \in \mathcal{A}_\circ \quad \text{for each } \lambda \in \mathbb{R}\setminus\{0\}.$$

By (5.47) for each $i \in \{1,2,\cdots,k\}$

$$\pi\left(B_i\right) = \lim_{\lambda\to\infty} \pi\left(B_i(\lambda)\right) \ \text{in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi). \tag{5.48}$$

This yields

$$\pi\left(B\right) = \lim_{\lambda\to\infty} \pi\left(B(\lambda)\right) \ \text{in the strong operator topology of } \mathfrak{B}(\mathscr{H}_\pi). \tag{5.49}$$

Therefore $\{B(\lambda) \in \mathcal{A}_\circ;\ \lambda \in \mathbb{R}\setminus\{0\}\}$ gives a net that converges to B.

We know that $\pi_\omega(\mathcal{A}_\circ)$ is dense in $\pi_\omega(\mathcal{F})$ in the strong operator topology of $\mathfrak{B}(\mathscr{H}_\omega)$, if ω is a regular state. This obviously implies that the subspace $\mathscr{H}_{\omega\,\circ}$ is dense in $\mathscr{H}_\omega$. $\square$

Remark 5.9. We have given a different treatment for Lemma 5.8 in [3].

5.4 Supersymmetry formula on an extended fermion-boson system

In the preceding subsection, we have defined an hermite superderivation δ_{s} and a one-parameter automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ on the fermion-boson C^*-system $\mathcal{F}$. The superderivation δ_{s} generates a super-transformation, while $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ describes time development. We aim to encode supersymmetry dynamics on $\mathcal{F}$ in terms of them. However, it is not a routine, since unbounded boson fields are not in the C^*-system $\mathcal{F}$, and accordingly the supersymmetry formula can not be written straightforwardly within $\mathcal{F}$. This subsection is devoted to some preparation to give an intermediate step for this aim.

We will introduce the following extended fermion-boson system as an auxiliary tool:

$$\mathcal{E} := \ast\text{-alg}\big\{c(f),\ R(\lambda,\, f),\ j(f) \equiv i1 - R(1,\, f)^{-1} \mid f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}),\ \lambda \in \mathbb{R}\setminus\{0\}\big\}. \tag{5.50}$$

The grading automorphism on $\mathcal{E}$ is given by

$$\gamma\big(c(f)\big) = -c(f) \text{ for all } f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}).$$

As defined previously in (3.1),

$$\mathcal{E}_{\mathrm{bos}}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big) \equiv \ast\text{-alg}\big\{R(\lambda,\, f),\ j(f) \mid f \in \mathscr{S}(\mathbb{R},\, \mathbb{R}),\ \lambda \in \mathbb{R}\setminus\{0\}\big\}. \tag{5.51}$$

We note that $\mathcal{E}$ and $\mathcal{E}_{\mathrm{bos}}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big)$ are formal algebras satisfying the algebraic relations given so far. Note that

$$\mathrm{Cliff}_0\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\big) \subset \mathcal{E} \supset \mathcal{E}_{\mathrm{bos}}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big). \tag{5.52}$$

The fermion subalgebra $\mathrm{Cliff}_0\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \tau\big)$ and the boson subalgebra $\mathcal{E}_{\mathrm{bos}}\big(\mathscr{S}(\mathbb{R},\, \mathbb{R}),\, \sigma\big)$ commute algebraically.

We shall embed $\mathcal{F}_0$ into $\mathcal{E}$ as a γ-invariant $\ast$-subalgebra. (Recall that $\mathcal{F}_0$ is the subalgebra of the fermion-boson C^*-system $\mathcal{F}$ defined in (5.9).) Therefore we have the following inclusions:

$$\mathcal{A}_\circ \subset \mathcal{F}_0 \subset \mathcal{E}. \tag{5.53}$$

We extend the hermite superderivation $\delta_{\mathrm{s}} : \mathcal{A}_\circ \to \mathcal{F}_0$ given in (5.29) to the extended graded system $\mathcal{E}$. For every $f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$ and $\lambda \in \mathbb{R}\setminus\{0\}$ set

$$\overline{\delta_{\mathrm{s}}}(c(f)) := j(f), \quad \overline{\delta_{\mathrm{s}}}(j(f)) := ic(f'), \quad \overline{\delta_{\mathrm{s}}}(R(\lambda,\, f)) := ic(f')R(\lambda,\, f)^2. \tag{5.54}$$

These are consistent and uniquely determine an hermite superderivation $\overline{\delta_{\mathrm{s}}}$ on the graded $\ast$-algebra $\mathcal{E}$, see [1].

We define a derivation on $\mathcal{E}$ that corresponds to the infinitesimal generator of the automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ give in Sect.5.3. (This will be rigorously verified later in Proposition 5.20.) For every $f \in \mathscr{S}(\mathbb{R},\, \mathbb{R})$ and $\lambda \in \mathbb{R}\setminus\{0\}$ set

$$\overline{d_0}\big(c(f)\big) := ic(f'), \quad \overline{d_0}\big(j(f)\big) := ij(f'), \quad \overline{d_0}(R(\lambda,\, f)) := iR(\lambda,\, f)j(f')R(\lambda,\, f). \tag{5.55}$$

These are consistent and uniquely determine an derivation on $\mathcal{E}$, see [1].

Remark 5.10. We will not extend the automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ to $\mathcal{E}$. The time development for the boson fields will be given through the GNS representation of regular states.

Remark 5.11. We shall indicate some notation differences between ours and those in [1]. Our $\mathcal{F}$ of (5.4) and $\mathcal{F}_0$ of (5.9) are denoted by $\mathcal{A}$ and $\mathcal{A}_0$, respectively in [1]. The hermite superderivation δ_{s} given in (5.29) and the derivation $\overline{d_0}$ in (5.55) are denoted by δ and δ_0, respectively in [1]; the subscript of δ_{s} indicates 'symmetric' with respect to the $*$-operation on superderivations as in (5.30). The domain algebra $\mathcal{A}_\circ$ of δ_{s} given in (5.28) is denoted by $\mathcal{D}_S$ in [1]. $\mathcal{E}$ defined in (5.50) is same in [1].

We will verify that $\overline{\delta_{\mathrm{s}}}$ is an extension of δ_{s} and that $\overline{\delta_{\mathrm{s}}}$ and $\overline{d_0}$ give an exact supersymmetry relation on $\mathcal{E}$.

Proposition 5.12. *The superderivation δ_{s} coincides with $\overline{\delta_{\mathrm{s}}}$ restricted to $\mathcal{A}_\circ$,*

$$\delta_{\mathrm{s}} = \overline{\delta_{\mathrm{s}}} \quad on \quad \mathcal{A}_\circ. \tag{5.56}$$

The following supersymmetry relation is satisfied on $\mathcal{E}$:

$$\overline{d_0} = \overline{\delta_{\mathrm{s}}}^2 \quad on \quad \mathcal{E}. \tag{5.57}$$

Proof. We will verify (5.56). We only have to check the first relation of (5.29). For every $f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$,

$$\begin{aligned}
\overline{\delta_{\mathrm{s}}}(\zeta(f)) &= \overline{\delta_{\mathrm{s}}}\left(c(f)R(1, f)\right) \\
&= \overline{\delta_{\mathrm{s}}}\left(c(f)\right) R(1, f) - c(f)\overline{\delta_{\mathrm{s}}}\left(R(1, f)\right) \\
&= j(f)R(1, f) - c(f)ic(f')R(1, f)^2 \\
&= (i1 - R(1, f)^{-1})R(1, f) - ic(f)c(f')R(1, f)^2 \\
&= iR(1, f) - 1 - ic(f)c(f')R(1, f)^2 \\
&= \delta_{\mathrm{s}}(\zeta(f)).
\end{aligned}$$

Next we will verify (5.57). For every $f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ we see

$$\begin{aligned}
\overline{\delta_{\mathrm{s}}} \cdot \overline{\delta_{\mathrm{s}}}\big(c(f)\big) &= \overline{\delta_{\mathrm{s}}}\big(j(f)\big) = ic(f') = \overline{d_0}\big(c(f)\big), \\
\overline{\delta_{\mathrm{s}}} \cdot \overline{\delta_{\mathrm{s}}}\big(j(f)\big) &= \overline{\delta_{\mathrm{s}}}\big(ic(f)\big) = ij(f') = \overline{d_0}\big(j(f)\big).
\end{aligned}$$

For every $f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}
\overline{\delta_{\mathrm{s}}} \cdot \overline{\delta_{\mathrm{s}}}\left(R(\lambda, f)\right) &= \overline{\delta_{\mathrm{s}}}\left(ic(f')R(\lambda, f)^2\right) = \overline{\delta_{\mathrm{s}}}\left(iR(\lambda, f)c(f')R(\lambda, f)\right) \\
&= iR(\lambda, f)\overline{\delta_{\mathrm{s}}}\left(c(f')\right) R(\lambda, f) + i\overline{\delta_{\mathrm{s}}}\left(R(\lambda, f)\right) c(f')R(\lambda, f) - iR(\lambda, f)c(f')\overline{\delta_{\mathrm{s}}}\left(R(\lambda, f)\right) \\
&= iR(\lambda, f)j(f')R(\lambda, f) + i\Big(ic(f')R(\lambda, f)^2c(f')R(\lambda, f) - R(\lambda, f)c(f')ic(f')R(\lambda, f)^2\Big) \\
&= iR(\lambda, f)j(f')R(\lambda, f) - 2c(f')^2\left(R(\lambda, f)^3 - R(\lambda, f)^3\right) \\
&= iR(\lambda, f)j(f')R(\lambda, f) = \overline{d_0}(R(\lambda, f)).
\end{aligned}$$

As both $\overline{d_0}$ and $\overline{\delta_{\mathrm{s}}}^2$ are derivations on $\mathcal{E}$, we can extend the above equalities to the whole $\mathcal{E}$. We thus obtain (5.57). ☐

Lemma 5.13. *The following identity in $\mathcal{F}_0$ holds*:

$$M_{A,\lambda}\delta_{\mathrm{s}}^2(A) = M_{A,\lambda}\overline{\delta_{\mathrm{s}}}^2(A) \quad for\ every\ A \in \mathcal{A}_\circ. \tag{5.58}$$

Proof. By (5.56) of Proposition 5.12 we can replace $\delta_{\rm s}$ appeared on the right-hand side of (5.34) by $\overline{\delta_{\rm s}}$. Hence for each $A \in \mathcal{A}_\circ$

$$\begin{aligned} M_{A,\lambda}\delta_{\rm s}^2(A) &= \delta_{\rm s}\left(M_{A,\lambda}\delta_{\rm s}(A)\right) - \delta_{\rm s}(M_{A,\lambda})\delta_{\rm s}(A) \\ &= \overline{\delta_{\rm s}}\left(M_{A,\lambda}\overline{\delta_{\rm s}}(A)\right) - \overline{\delta_{\rm s}}(M_{A,\lambda})\overline{\delta_{\rm s}}(A) \\ &= \overline{\delta_{\rm s}}(M_{A,\lambda})\overline{\delta_{\rm s}}(A) + M_{A,\lambda}\overline{\delta_{\rm s}}\left(\overline{\delta_{\rm s}}(A)\right) - \overline{\delta_{\rm s}}(M_{A,\lambda})\overline{\delta_{\rm s}}(A) \\ &= M_{A,\lambda}\overline{\delta_{\rm s}}^2(A), \end{aligned}$$

where we have used the graded Leibniz rule of $\overline{\delta_{\rm s}}$ and $M_{A,\lambda} \in \mathcal{R}_0\big(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma\big) \subset \mathcal{E}_+$. $\square$

We will show the commutativity between the fermion system $\mathrm{Cliff}_0\big(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau\big)$ and the unbounded boson system $\mathcal{E}_{\rm bos}\big(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma\big)$ as follows.

Lemma 5.14. *Let ω be an arbitrary regular state of $\mathcal{F}$, and let $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ denote its GNS representation. Assume that*

$$\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\big) \quad \textit{for all } f_1, \cdots, f_n \in X,\ n \in \mathbb{N}.$$

Then on the (dense) subspace $\pi_\omega(\mathcal{F}_0)\,\Omega_\omega$ of $\mathscr{H}_\omega$, any operator in $\pi_\omega\left(\mathrm{Cliff}_0\big(\mathscr{S}(\mathbb{R}, \mathbb{R}), \tau\big)\right)$ and any operator in $\pi_\omega\left(\mathcal{E}_{\rm bos}\big(\mathscr{S}(\mathbb{R}, \mathbb{R}), \sigma\big)\right)$ commute with each other.

Proof. By Proposition 5.1 the assumption implies

$$\pi_\omega(\mathcal{F}_0)\,\Omega_\omega \subset \mathrm{Dom}\big(j_\pi(f_1)\cdots j_\pi(f_n)\big) \quad \text{for all } f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}.$$

By applying Corollary 5.5 to the present case we obtain the desired commutativity relation. $\square$

Proposition 5.15. *Let ω be an arbitrary regular state of $\mathcal{F}$, and let $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ denote its GNS representation. Assume that*

$$\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\big) \quad \textit{for all } f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}.$$

Then the state ω on $\mathcal{F}$ can be extended to the $*$-algebra $\mathcal{E}$ by determining

$$\begin{aligned} &\omega\left(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)j(f_1)\cdots j(f_n)\right) \\ &:= \Big(\Omega_\omega,\ \pi_\omega\left(c(h_1)\cdots c(h_k)R(\mu_1, g_1)\cdots R(\mu_m, g_m)\right) j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\Omega_\omega\Big) \end{aligned}$$

for all $h_1, \cdots, h_k,\ g_1, \cdots, g_m,\ f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ \mu_1, \cdots, \mu_m \in \mathbb{R}\setminus\{0\},\ k, m, n \in \mathbb{N}$. Similarly the GNS representation $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ extends to $\mathcal{E}$ by setting

$$\pi_\omega\big(j(f)\big) := j_{\pi_\omega}(f) \quad \textit{for every } f \in \mathscr{S}(\mathbb{R}, \mathbb{R}).$$

Proof. The statement can be shown in the same way as Proposition 3.1 owing to Proposition 5.1 and Corollary 5.5 (or Lemma 5.14). $\square$

Let us generalize Definition 4.1 to the fermion-boson system $\mathcal{F}$ to specify the states under consideration.

Definition 5.16. Let $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let ω be a regular state on $\mathcal{F}$ whose GNS representation is denoted as $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$. Assume that the conditions (4.1) and (4.2) in Definition 4.1 are saturated, namely,

$$\Omega_\omega \in \mathrm{Dom}\big(j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\big) \quad \text{for all } f_1, \cdots, f_n \in \mathscr{S}(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}, \tag{5.59}$$

and

$$\lim_{t\to 0}\frac{j_{\pi_\omega}(f_t)-j_{\pi_\omega}(f)}{t}\Omega_\omega=-j_{\pi_\omega}\left(f'\right)\Omega_\omega \quad \text{for every } f\in\mathscr{S}(\mathbb{R},\,\mathbb{R}), \tag{5.60}$$

where the limit on the left-hand side is taken with respect to the strong topology of $\mathscr{H}_\omega$. The set of all regular states on $\mathcal{F}$ satisfying (5.59) and (5.60) is denoted by $K_{\alpha_t}(\mathcal{F})$.

As we have anticipated before, we have to justify that $\overline{d_0}$ is the infinitesimal generator of the automorphism group $\{\alpha_t\in \mathrm{Aut}\mathcal{F}\,;\ t\in\mathbb{R}\}$. This will be done in the following.

Lemma 5.17. *Let $\{\alpha_t\in \mathrm{Aut}\mathcal{F}\,;\ t\in\mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let $\omega\in K_{\alpha_t}(\mathcal{F})$ of Definition* 5.16*. Let $\left(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\right)$ denote the GNS representation for ω. Then for every $f\in\mathscr{S}(\mathbb{R},\,\mathbb{R})$,*

$$\frac{d}{dt}j_{\pi_\omega}(f_t)\xi\,\Big|_{t=0}=-j_{\pi_\omega}\left(f'\right)\xi \quad \textit{for each}\ \ \xi\in\pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega, \tag{5.61}$$

where the derivation on the left hand side is with respect to the strong topology of $\mathscr{H}_\omega$.

Proof. By noting Lemma 5.14 we can show the statement in the same way as Lemma 4.3. □

Proposition 5.18. *Let $\{\alpha_t\in \mathrm{Aut}\mathcal{F}\,;\ t\in\mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let $\omega\in K_{\alpha_t}(\mathcal{F})$ of Definition* 5.16*. Then for every $f\in\mathscr{S}(\mathbb{R},\,\mathbb{R})$,*

$$\frac{d}{dt}\pi_\omega\left(\alpha_t\left(R(\lambda,\,f)\right)\right)\xi\Big|_{t=0}=-\pi_\omega(R(\lambda,\,f))j_{\pi_\omega}(f')\pi_\omega(R(1,\,f))\xi \quad \textit{for each}\ \ \xi\in\pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega, \tag{5.62}$$

where the derivation on the left-hand side is with respect to the strong topology of $\mathscr{H}_\omega$.

Proof. This follows from Proposition 4.4 and Lemma 5.14. □

Remark 5.19. In the proof of Lemma 5.17 and also in Proposition 5.18 the fermion-boson commutativity (shown in Lemma 5.14 Corollary 5.5) is used crucially.

Proposition 5.20. *Let $\{\alpha_t\in \mathrm{Aut}\mathcal{F}\,;\ t\in\mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let $\overline{d_0}$ denote the derivation on the extended system $\mathcal{E}$ defined in* (5.55)*. Let ω be an arbitrary state of $K_{\alpha_t}(\mathcal{F})$ of Definition* 5.16*. Let $\left(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\right)$ denote the GNS representation for ω which is extended to $\mathcal{E}$ by Proposition* 5.15*. Then for every $A\in\mathcal{F}_0$,*

$$\frac{d}{dt}\pi_\omega\left(\alpha_t\left(A\right)\right)\xi\Big|_{t=0}=i\pi_\omega\big(\overline{d_0}(A)\big)\xi \quad \textit{for every}\ \ \xi\in\pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega, \tag{5.63}$$

where the derivation on the left-hand side is in the strong topology of $\mathscr{H}_\omega$. Specifically, for an arbitrary monomial element $A=c(h_1)\cdots c(h_k)R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\in\mathcal{F}_0$ with $h_1,\cdots,h_k,\,g_1,\cdots,g_m\in\mathscr{S}(\mathbb{R},\,\mathbb{R})$, $\mu_1,\cdots,\mu_m\in\mathbb{R}\setminus\{0\}$, $k,m\in\mathbb{N}$,

$$\begin{aligned}
&\frac{d}{dt}\pi_\omega\Big(\alpha_t\big(c(h_1)\cdots c(h_k)R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\big)\Big)\xi\Big|_{t=0}\\
&=-\sum_{i=1}^{k}\pi_\omega\left(c(h_1)\cdots c({h_i}')\cdots c(h_k)R(\mu_1,\,g_1)\cdots R(\mu_m,\,g_m)\right)\xi\\
&-\sum_{j=1}^{m}\pi_\omega\left(c(h_1)\cdots c(h_k)R(\mu_1,\,g_1)\cdots\pi_\omega(R(1,\,g_j))j_{\pi_\omega}({g_j}')\pi_\omega(R(1,\,g_j))\cdots R(\mu_m,\,g_m)\right)\xi.
\end{aligned} \tag{5.64}$$

Proof. As $\omega \in K_{\alpha_t}(\mathcal{F})$, both (5.41) of Lemma 5.6 and (5.62) of Proposition 5.18 hold. From these we obtain (5.64) by the Leibniz rule for derivations. By comparing (5.64) with the definition of $\overline{d_0}$ (5.55) we can see that (5.63) is satisfied. $\square$

Remark 5.21. As $\overline{d_0}(\mathcal{F}_0) \not\subset \mathcal{F}$, even $\overline{d_0}(\mathcal{A}_\circ) \not\subset \mathcal{F}$ due to $\overline{d_0}(R(\lambda,\, f)) = iR(\lambda,\, f)j(f')R(\lambda,\, f) \notin \mathcal{F}$, extension of the GNS representation π_ω to $\mathcal{E}$ is required on the right-hand side of the formula (5.63).

The following statement is the goal of this subsection. This is a supersymmetry formula on the dense algebra $\mathcal{F}_0$ given with help of the extended system $\mathcal{E}$.

Theorem 5.22. *Let $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let $\overline{\delta_{\mathrm{s}}}$ denote the hermite superderivation and $\overline{d_0}$ denote the derivation on the extended fermion-boson system $\mathcal{E}$ given in* (5.54) *and* (5.55)*, respectively. Suppose that ω is an arbitrary state of $K_{\alpha_t}(\mathcal{F})$ of Definition* 5.16*. Then for every $A \in \mathcal{F}_0$*

$$-i\frac{d}{dt}\pi_\omega\big(\alpha_t(A)\big)\xi\Big|_{t=0} = \pi_\omega\big(\overline{d_0}(A)\big)\xi = \pi_\omega\big(\overline{\delta_{\mathrm{s}}}^2(A)\big)\xi \quad \textit{for every} \ \ \xi \in \pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega, \tag{5.65}$$

where the derivation is with respect to the strong topology of $\mathscr{H}_\omega$, and the GNS representation $\big(\pi_\omega,\ \mathscr{H}_\omega,\ \Omega_\omega\big)$ for ω is extended to $\mathcal{E}$ by Proposition 5.15.

Proof. By combining (5.57) in Proposition 5.12 with (5.63) in Proposition 5.20 we obtain (5.65). $\square$

5.5 Supersymmetry formula on the fermion-boson C^*-system

In the preceding subsection, we have given a supersymmetry formula using essentially the extended fermion-boson system $\mathcal{E}$ into which the dense subalgebra $\mathcal{F}_0$ of $\mathcal{F}$ is embedded. We aim to provide a supersymmetry formula with the same dynamics content *within* the fermion-boson C^*-system $\mathcal{F}$. (So we are not allowed to use either $\overline{\delta_{\mathrm{s}}}$ or $\overline{d_0}$ defined on $\mathcal{E}$. We can use only the superderivation δ_{s} of $\mathcal{F}$, the one-parameter automorphism group $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$, and some states on $\mathcal{F}$.)

Theorem 5.23. *Let $\{\alpha_t \in \mathrm{Aut}\mathcal{F}\,;\ t \in \mathbb{R}\}$ denote the quasi-free automorphism group of $\mathcal{F}$ associated with the shift translations. Let δ_{s} denote the hermite superderivation defined on $\mathcal{A}_\circ$ as in* (5.29)*. Let ω denote an arbitrary state of $K_{\alpha_t}(\mathcal{F})$ of Definition* 5.16*. Then for every $A \in \mathcal{A}_\circ$*

$$-i\frac{d}{dt}\pi_\omega\big(M_{A,\lambda}\alpha_t(A)\big)\xi\Big|_{t=0} = \pi_\omega\big(M_{A,\lambda}\delta_{\mathrm{s}}^2(A)\big)\xi \quad \textit{for every} \ \ \xi \in \pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega, \tag{5.66}$$

where $M_{A,\lambda}\delta_{\mathrm{s}}^2(A) \in \mathcal{F}_0$ is given in (5.34).

Proof. The formula (5.65) in Theorem 5.22 implies that for every $A \in \mathcal{A}_\circ \subset \mathcal{F}_0$

$$-i\frac{d}{dt}\pi_\omega\big(M_{A,\lambda}\alpha_t(A)\big)\xi\Big|_{t=0} = \pi_\omega\big(M_{A,\lambda}\overline{\delta_{\mathrm{s}}}^2(A)\big)\xi \ \ \text{for every} \ \ \xi \in \pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega.$$

By this and (5.58) in Lemma 5.13 we have (5.66). $\square$

Remark 5.24. A weak infinitesimal supersymmetry formula analogous to Theorem 5.23 is given in [1, Theorem.5.8]: For each $A \in \mathcal{A}_\circ$

$$-i\frac{d}{dt}\varphi\big(CM_{A,\lambda}\alpha_t(A)B\big)\Big|_{t=0} = \varphi\big(CM_{A,\lambda}\delta_{\mathrm{s}}^2(A)B\big) \ \ \text{for all} \ \ B, C \in \mathcal{F}_0. \tag{5.67}$$

In this statement φ is an unbounded supersymmetric quasi-free (graded-KMS) functional. It is rather straightforward to see that the formula (5.67) holds for the supersymmetric state which is given as a product Fock state, see [3]. However, its generalization to non-supersymmetric states does not follow from the original proof of [1, Theorem.5.8]. The extension to any regular state which is not necessarily either supersymmetric or quasi-free is our achievement.

Remark 5.25. From the inclusion $\mathcal{A}_\circ \subset \mathcal{F}_0$ and the formula (5.66) of Theorem 5.23 it follows obviously that for every $A \in \mathcal{A}_\circ$

$$-i\frac{d}{dt}\pi_\omega\big(M_{A,\lambda}\alpha_t(A)\big)\xi\Big|_{t=0} = \pi_\omega\big(M_{A,\lambda}\delta_{\mathrm{s}}^2(A)\big)\xi \ \text{ for every } \ \xi \in \pi_\omega\left(\mathcal{A}_\circ\right)\Omega_\omega. \tag{5.68}$$

Recall that the smaller subspace $\mathscr{H}_{\omega\,\circ} = \pi_\omega\left(\mathcal{A}_\circ\right)\Omega_\omega$ used in the above formula is dense in $\mathscr{H}_\omega$ as shown in Lemma 5.8. Furthermore, if $\omega \in K_{\alpha_t}(\mathcal{F})$ is supersymmetric, i.e. invariant under δ_{s}, then the supercharge operator implementing δ_{s} upon the GNS Hilbert space $\mathscr{H}_\omega$ is essentially self-adjoint on $\mathscr{H}_{\omega\,\circ}$. See [3] for the detail.

Corollary 5.26. *Assume the same setting as in Theorem* 5.23. *Then for every* $A \in \mathcal{A}_\circ$:

$$-i\frac{d}{dt}\pi_\omega\big(\alpha_t(A)\big)\xi\Big|_{t=0} = \lim_{\lambda\to\infty}\pi_\omega\big(M_{A,\lambda}\delta_{\mathrm{s}}^2(A)\big)\xi \ \textit{ for each } \ \xi \in \pi_\omega\left(\mathcal{F}_0\right)\Omega_\omega. \tag{5.69}$$

Proof. Taking the limit $\lambda \to \infty$ of (5.66) yields (5.69) due to (5.32). □

6 Summary and concluding remarks

We have shown some general properties of regular states (which are not necessarily quasi-free) and quasi-free dynamics on the resolvent algebra. Making use of those fundamental results we have investigated supersymmetric quasi-free dynamics.

We have focused on a simplified supersymmetry model by Buchholz-Grundling. It is defined on the C^* tensor-product $\mathcal{F}$ of a Clifford algebra and a resolvent algebra. This supersymmetry model can be straightforwardly formulated on the extended fermion-boson system $\mathcal{E}$ as in Proposition 5.12. Recall that $\mathcal{E}$ is the polynomial algebra generated by Clifford elements, resolvent elements and unbounded boson field operators. This formal relation does not take analytical aspects of the dynamics into account. (Proposition 5.12 is from [1], it is not our result.)

In Theorem 5.22 we formulate the supersymmetry dynamics on the norm dense $*$-subalgebra $\mathcal{F}_0$. This is given as an operator equality on the GNS Hilbert space for an arbitrary state of $K_{\alpha_t}(\mathcal{F})$. Recall that $\mathcal{F}_0$ is the subalgebra of $\mathcal{F}$ generated by all the polynomials of Clifford elements and resolvent elements and that $\mathcal{F}_0$ is embedded into the extended fermion-boson system $\mathcal{E}$. $K_{\alpha_t}(\mathcal{F})$ denotes the set of regular states of $\mathcal{F}$ satisfying some natural assumption as defined in Definition 5.16.

In Theorem 5.23 we formulate the supersymmetry dynamics on $\mathcal{A}_\circ$, i.e. the domain of the superderivation. This is given as an operator equality on the GNS Hilbert space for an arbitrary state of $K_{\alpha_t}(\mathcal{F})$. Let us compare Theorem 5.23 with Theorem 5.22. While Theorem 5.22 makes use of the extended system $\mathcal{E}$, the formulation of Theorem 5.23 is given completely within $\mathcal{F}$. Note that $\mathcal{A}_\circ$ is not norm dense in the total system $\mathcal{F}$, however, it is dense in $\mathcal{F}$ in a weaker sense as in Lemma 5.8.

Let us explain some merits provided by Theorem 5.22 and Theorem 5.23. First, both theorems are based on the state space $K_{\alpha_t}(\mathcal{F})$. It includes not only the supersymmetric vacuum state but also non-supersymmetric states. It is not restricted to quasi-free states. It seems that $K_{\alpha_t}(\mathcal{F})$ is large and natural in terms of physics. See *Remark* 4.2. By this

fact, we may assert that the given supersymmetric dynamics is a C^*-dynamics; it is not merely a W^*-dynamics induced by some special representation. We refer to [5, Section 4.8] for some related issue on how C^*-dynamics for quantum field models (continuous models in the terminology used there) would be formulated. Second, both theorems are given by the operator equality on the dense subspace (which is generated by acting $\mathcal{F}_0$ on the GNS cyclic vector), not by expectation values as in [1]. We have shown in [3] that the supercharge operator upon the GNS Hilbert space is essentially self-adjoint on the dense subspace generated by the domain $\mathcal{A}_\circ$ of the superderivation on the cyclic vector if the regular state is supersymmetric. We may assert that the superderivation can encode the dynamics fully, even though its domain $\mathcal{A}_\circ$ is not norm dense in the total system.

The supersymmetry model which we have considered in this note is extremely simplified, as it is originally intended for investigation on model-independent aspects of supersymmetry in a C^*-algebra, see [1]. It would be interesting to construct other standard supersymmetry models in the C^*-algebraic framework. We conjecture that some results given in this note will be generalized to such general situations, though, it is not always a routine.

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